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Bound states in the continuum in a single-level Fano-Anderson model

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Abstract. Bound states in the continuum (BIC) are shown to exist in a single-level Fano-Anderson model with a colored interaction between the discrete state and a structured tight-binding continuum, which may describe mesoscopic electron or photon transport in a semi-infinite one-dimensional lattice. The existence of BIC is explained in the lattice realization as a boundary effect induced by lattice truncation.

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1 Introduction

Since the pioneering proposal of the existence of isolated quantum mechanical bound states embedded in the continuum, made by von Neumann and Wigner [1] in the study of the one-particle Schrödinger equation with certain spatially oscillating attractive potentials, several theoretical and a few experimental studies have demonstrated the existence of "bound states in the continuum" (BIC) in a wide range of different physical systems [2–22]. BIC may be found in certain atomic or molecular systems [2,5,16], such as hydrogen atom in a uniform magnetic field [5], in semiconductor superlattice structures [3,4,8,11,18], in mesoscopic electron transport and quantum waveguides [6,7,9,10,12,14,15], and in quantum dot systems [13,17,19–21]. The question of existence of BIC has been also addressed for the famous Fano-Anderson Hamiltonian [23–25] (also referred to as the Friedrichs-Lee Hamiltonian [26–28]), which model the process of quantum mechanical decay of an unstable localized state coupled to a continuum in different contexts such as atomic physics, quantum electrodynamics, solid state and high-energy physics (see, e.g., [25,29–32]). BIC in Fano-Anderson-like models are commonplace in case where *several* (i.e. more than one) discrete states are coupled to a common continuum; a noteworthy example of this case in condensed-matter physics is provided, for instance, by quantum transport and scattering in dot molecules attached to leads [17,19–22]. In those systems the existence of BIC is usually related to the destruction of discrete-continuum decay channels via quantum interference through a typical trapping mechanism. The conditions for the existence of BIC for a general multi-level Fano-Anderson Hamiltonian have been recently stated in reference [33]; in particular, a sufficient condition that ensures the *non-existence* of BIC has been demonstrated. In case of a *single* localized state embedded in and interacting with a continuum, the existence of bound states (sometimes referred to as "dressed bound states") has been acknowledged on many occasions and related to threshold effects or to singularities or gaps in the density of states of the continuum (see, e.g., [25,31,34–39]). However, such dressed bound states have usually an energy *outside* the continuum [25]: effects such as fractional decay, population trapping and atom-photon bound states found in several physical models describing the decay of a single discrete level coupled to a continuum [31,37–39] are in fact most of the times related to the existence of dressed bound states with an energy *outside* the continuum. Conversely, BIC have seldom been encountered in the single-level Fano-Anderson model: Examples of BIC involving a single localized state coupled with a specially-structured continuum have been found in the study of certain exactly-solvable electrodynamic models of spontaneous emission decay with a density of modes showing a point-like gap [40,41], however physical realizations of such models were not proposed [31].

It is the aim of this work to present an exactly-solvable Fano-Anderson Hamiltonian with a single discrete level coupled to a structured continuum by a colored interaction which supports BIC. The basic idea, here, is to engineer the spectral coupling between the discrete state and the continuum in such a way to suppress the interaction between the discrete level and one (or more) frequencies of the continuum (see [33]). This procedure is therefore rather distinct than engineering the density of states of

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the continuum itself to produce point-like gaps as in references [40,41]. The proposed model of a colored interaction is of physical relevance to certain condensed-matter and photonic systems, since it may describe the charge transfer dynamics of adatoms to a semi-infinite one-dimensional lattice of quantum dots [42] or photon tunneling dynamics in semi-infinite optical waveguide arrays or coupled optical resonators [43–48] within the tight-binding approximation. In these lattice realizations, the tight-binding analysis allows one to simply explain the appearance of BIC as a *boundary effect* due to truncation of the lattice.

The paper is organized as follows. In Section 2, the Fano-Anderson model describing the coupling of a single localized state with a continuum is briefly reviewed, and the conditions for the existence of bound states either inside or outside the continuum are presented. Section 3 proposes an exactly-solvable model with colored interaction which supports BIC; exact analytical results for the fractional decay induced by BIC are also presented. Finally, in Section 4 a lattice realization of the model is presented, together with a simple physical explanation for the existence of BIC.

2 Bound states and decay dynamics for the single-level Fano-Anderson model

2.1 Basic model

The starting point of the analysis is provided by a standard Fano-Anderson model describing the interaction of a discrete state $|a\rangle$, of energy $\hbar\omega_a$, with a continuum described by a set of continuous states $|k\rangle$ with energy $\hbar\omega(\kappa)$ (see, for instance, [25,29,31]). The Hamiltonian of the interacting system can be written as $H = H_0 + V$, where

$$H_0 = \hbar\omega_a|a\rangle\langle a| + \int dk\ \hbar\omega(k)|k\rangle\langle k| \tag{1}$$

is the Hamiltonian of the non-interacting discrete and continuous states, and

$$V = \hbar \int dk\, [v(k)|a\rangle\langle k| + v^*(k)|k\rangle\langle a|] \tag{2}$$

is the interaction part. Normalization has been assumed such that $\langle a|a\rangle = 1$, $\langle a|k\rangle = 0$ and $\langle k'|k\rangle = \delta(k - k')$. If we expand the wave function $|\psi\rangle$ of the system as $|\psi\rangle = c_a(t)|a\rangle + \int dkc(k,t)|k\rangle$, the expansion coefficients $c_a(t)$ and $c(k,t)$ satisfy the coupled-mode equations

$$i\dot{c}_a(t) = \omega_a c_a + \int dkv(k)c(k,t) \tag{3}$$

$$i\dot{c}(k,t) = \omega(k)c(k,t) + v^*(k)c_a(t), \tag{4}$$

where the dot indicates the derivative with respect to time. Typically, we assume that the frequency $\omega(k)$ of continuous states spans a finite interval (a band) $\omega_1 < \omega < \omega_2$ for the allowed values of the continuous variable k, and that the frequency ω_a of the discrete level is embedded in the continuum, i.e. that $\omega_1 < \omega_a < \omega_2$.

2.2 Bound states

The eigenstates $|\psi_E\rangle$ of H corresponding to the eigenvalue $E = \hbar\Omega$ are obtained from the eigenvalue equation $H|\psi_E\rangle = \hbar\Omega|\psi_E\rangle$. After introduction of the density of states $\rho(\omega) = \partial k/\partial\omega$, from equations (1) and (2) it follows that the eigenfrequencies Ω are found as the eigenvalues of the system

$$\Omega c_a = \omega_a c_a + \int_{\omega_1}^{\omega_2} d\omega \sqrt{\rho(\omega)}v(\omega)\tilde{c}(\omega) \tag{5}$$

$$\Omega\tilde{c}(\omega) = \omega\tilde{c}(\omega) + \sqrt{\rho(\omega)}v^*(\omega)c_a, \tag{6}$$

where we have set $\tilde{c}(\omega) = \sqrt{\rho(\omega)}c(\omega)$. As the continuous spectrum of H is the same as that of H_0, isolated eigenvalues corresponding to bound states may or may not occur for H. The energy E corresponds to a bound state of H provided that $|\psi_E\rangle$ is square integrable. This implies

$$\|\psi_E\|^2 = |c_a|^2 + \int_{\omega_1}^{\omega_2} d\omega|\tilde{c}(\omega)|^2 < \infty. \tag{7}$$

We have to distinguish two cases.

(i) *Bound states outside the continuum.* This is the most common case, which has been studied on several occasions. A bound state with energy $\hbar\Omega$ outside the continuum exists provided that a root of the equation

$$\Omega - \omega_a = \Delta(\Omega) \tag{8}$$

can be found outside the band (ω_1, ω_2), where we have set

$$\Delta(\Omega) = \int_{\omega_1}^{\omega_2} d\omega \frac{\rho(\omega)|v(\omega)|^2}{\Omega - \omega}. \tag{9}$$

The conditions for the existence of bound states of this kind have been extensively investigated in the literature (see, e.g., [31,38,39]). For instance, equation (8) admits always two solutions outside the interval (ω_1, ω_2) whenever $\rho(\omega)|v(\omega)|^2$ does not vanish at the edge of the band, since in this case $\Delta(\Omega)$ diverges to $\mp\infty$ as $\Omega \to \omega_{1,2}^{\mp}$.

(ii) *Bound states inside the continuum.* As shown in reference [33], a bound state at frequency Ω inside the continuum may exist provided that the following two conditions are simultaneously satisfied

$$|v(\Omega)|^2\rho(\Omega) = 0\ \ ,\ \ \Omega - \omega_a = \Delta(\Omega). \tag{10}$$

Additionally, $v(\omega)\sqrt{\rho(\omega)}$ should vanish as $\omega \to \Omega$ at least as $\sim (\omega - \Omega)$ in order to ensure a finite norm (Eq. (7)). The first equation in (10) can be satisfied for either $\rho(\Omega) = 0$ or $v(\Omega) = 0$. The former case, which corresponds to a point-like gap in the density of states inside the band, has been previously considered for some special density of state profiles [40,41], which however do not seem to have simple physical realizations [31]. The latter case, $v(\Omega) = 0$, implies that the discrete state $|a\rangle$ does not interact with the continuous state of frequency Ω, and thus implies a "colored" interaction profile $v(\omega)$ with one zero at $\omega = \Omega$. However, at such a frequency the additional condition $\omega_a = \Omega - \Delta(\Omega)$ must be simultaneously satisfied, which means that BIC may exist solely at a prescribed energy $\hbar\omega_a$ of the level $|a\rangle$.

2.3 Decay dynamics

Consider now the decay dynamics of the unstable state $|a\rangle$ embedded in the continuum. This corresponds to solving equations (3) and (4) with the initial conditions $c_a(0) = 1$ and $c(k,0) = 0$, which can be done by e.g. a Laplace transformation or a Green's function analysis (see, for instance, [25,29,31]). Indicating by $\hat{c}_a(s) = \int_0^\infty dt c_a(t) \exp(-st)$ [Re$(s) > 0$] the Laplace transform of $c_a(t)$, from equations (3) and (4) one obtains

$$\hat{c}_a(s) = \frac{i}{is - \omega_a - \Sigma(s)}, \tag{11}$$

and then, after inversion

$$c_a(t) = \frac{1}{2\pi} \int_{0^+ - i\infty}^{0^+ + i\infty} ds \frac{\exp(st)}{is - \omega_a - \Sigma(s)}, \tag{12}$$

where $\Sigma(s)$ is the self-energy, given by

$$\Sigma(s) = \int dk \frac{|v(k)|^2}{is - \omega(k)} = \int_{\omega_1}^{\omega_2} d\omega \frac{\rho(\omega)|v(\omega)|^2}{is - \omega}. \tag{13}$$

Possible poles on the imaginary axis of $\hat{c}_a(s)$ correspond to bound states of H and are responsible for fractional decay of the amplitude $c_a(t)$. In fact, using the property

$$\Sigma(s = -i\omega \pm 0^+) = \Delta(\omega) \mp i\pi\rho(\omega)|v(\omega)|^2 \tag{14}$$

with

$$\Delta(\omega) = \mathcal{P} \int_{\omega_1}^{\omega_2} d\omega' \frac{\rho(\omega')|v(\omega')|^2}{\omega - \omega'} \tag{15}$$

the poles $s_p = -i\Omega$ of $\hat{c}_a(s)$ satisfy the conditions $\Omega - \omega_a = \Delta(\Omega)$ and $\rho(\Omega)|v(\Omega)|^2 = 0$, i.e. they are located in correspondence of the bound states (either outside or inside the continuum) of H. In absence of poles (i.e. of bound states of H), $c_a(t)$ decays to zero, whereas in presence of poles equation (12) can be written as the sum of a contour (decaying) integral plus the (non-decaying) pole contributions.

3 BIC in a Fano-Anderson model with a colored interaction: an exactly-solvable model

In this section we present an exactly-solvable Fano-Anderson model, describing the interaction of a single discrete state with a continuum, which admits of BIC. Precisely, we assume the following colored interaction function

$$v(k) = \sqrt{\frac{2}{\pi}} \kappa_a \sin(n_0 k) \tag{16}$$

and the following tight-binding dispersion curve for the band of continuous states

$$\omega(k) = -2\kappa_0 \cos k \tag{17}$$

where $0 \leq k \leq \pi$, κ_0 and κ_a are positive real parameters, and n_0 is a positive and nonvanishing integer number, i.e. $n_0 = 1, 2, 3, ...$ A physical realization of this model will be described in the next section. We note that this model is a generalization of the well-known tight-binding Fano-Anderson model with a constant interaction coupling $v(k) = \text{const.}$, which is known to show bound states solely *outside* the continuum [25]. Note also that for this band model the density of states, given by

$$\rho(\omega) = \frac{\partial k}{\partial \omega} = \frac{1}{\sqrt{4\kappa_0^2 - \omega^2}}, \tag{18}$$

shows van-Hove singularities at the band edges $\omega = \pm 2\kappa_0$. For constant coupling [25], these singularities are responsible for the existence of two bound states outside the band from either sides. However, for the colored coupling considered in our case (Eq. (16)) one has

$$G(\omega) \equiv \rho(\omega)|v(\omega)|^2 = \frac{2\kappa_a^2}{\pi\sqrt{4\kappa_0^2 - \omega^2}} \sin^2 \left[n_0 \cos^{-1}\left(\frac{\omega}{2\kappa_0}\right) \right] \tag{19}$$

which vanishes at the band edge. From equations (13, 16) and (17), the self-energy $\Sigma(s)$ can be calculated in an exact form and reads

$$\begin{aligned} \Sigma(s) &= \frac{2\kappa_a^2}{\pi} \int_0^\pi dk \frac{\sin^2(n_0 k)}{is + 2\kappa_0 \cos k} \\ &= -\frac{i\kappa_a^2}{\sqrt{s^2 + 4\kappa_0^2}} \left[1 - \left(\frac{i\sqrt{s^2 + 4\kappa_0^2} - is}{2\kappa_0} \right)^{2n_0} \right]. \end{aligned} \tag{20}$$

Using equation (14), the following expression for $\Delta(\omega) = \text{Re}[\Sigma(s = -i\omega \pm 0^+)]$ can be then derived

$$\Delta(\omega) = \begin{cases} -\frac{\kappa_a^2}{\sqrt{\omega^2 - 4\kappa_0^2}} \left[1 - \left(\frac{\sqrt{\omega^2 - 4\kappa_0^2} + \omega}{2\kappa_0} \right)^{2n_0} \right], & \omega < -2\kappa_0 \\ \frac{\kappa_a^2}{\sqrt{4\kappa_0^2 - \omega^2}} \sin\left[2n_0 \cos^{-1}\left(\frac{\omega}{2\kappa_0} \right) \right], & |\omega| < 2\kappa_0 \\ \frac{\kappa_a^2}{\sqrt{\omega^2 - 4\kappa_0^2}} \left[1 - \left(\frac{\sqrt{\omega^2 - 4\kappa_0^2} - \omega}{2\kappa_0} \right)^{2n_0} \right], & \omega > 2\kappa_0. \end{cases} \tag{21}$$

The behavior of $G(\omega) \equiv \rho(\omega)|v(\omega)|^2$ (Eq. (19)) and $\Delta(\omega)$ (Eq. (21)) for increasing values of n_0 is shown in Figure 1. Note the oscillatory behavior of both $G(\omega)$ and $\Delta(\omega)$, with the existence of $(2n_0 - 1)$ zeros of $\Delta(\omega)$ at $\omega_l = -2\kappa_0 \cos[l\pi/(2n_0)]$ ($l = 1, 2, ..., 2n_0 - 1$) and of $(n_0 + 1)$ zeros of $G(\omega)$ at $\omega_m = -2\kappa_0 \cos(m\pi/n_0)$ ($m = 0, 1, 2, ..., n_0$). We can then specialize the general results of Section 2 to the present model.

(i) *Bound states outside the continuum.* At most two bound states at frequency Ω outside the band $(-2\kappa_0, 2\kappa_0)$ from either sides may exist. Precisely, a bound state at frequency $\Omega > 2\kappa_0$ exists provided that $2\kappa_0 - \omega_a < \Delta(2\kappa_0)$, i.e. $\omega_a > 2\kappa_0 - \kappa_a^2 n_0/\kappa_0$, whereas a bound state at frequency $\Omega < -2\kappa$ does exist for $-2\kappa_0 - \omega_a > \Delta(-2\kappa_0)$, i.e. for $\omega_a < -2\kappa_0 + \kappa_a^2 n_0/\kappa_0$. Therefore, if the frequency ω_a of the discrete level lies inside the interval

$$-1 + \frac{\kappa_a^2 n_0}{2\kappa_0^2} < \frac{\omega_a}{2\kappa_0} < 1 - \frac{\kappa_a^2 n_0}{2\kappa_0^2} \tag{22}$$

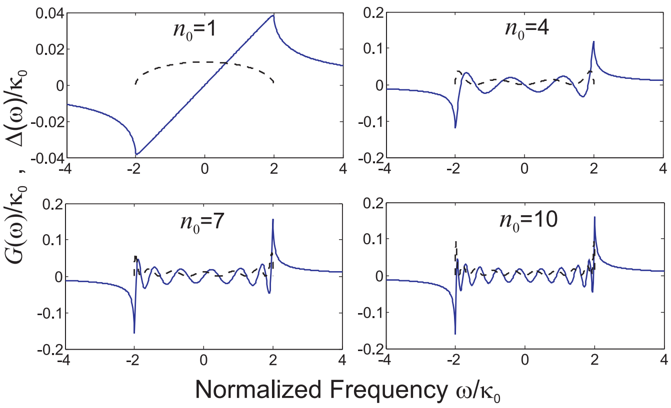

Fig. 1. Behavior of $G(\omega) = \rho(\omega)|v(\omega)|^2$ (dotted curves) and $\Delta(\omega)$ (solid curves), normalized to κ_0, versus normalized frequency ω/κ_0 for $(\kappa_a/\kappa_0) = 0.2$ and for increasing values of integer n_0.

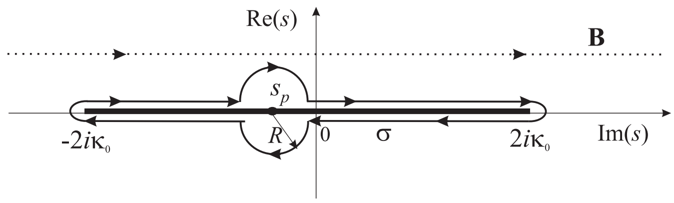

Fig. 2. Deformation of the Bromwich path B (dotted line) for inverse Laplace transformation. The bold solid curve is the branch cut, whereas the deformed path is represented by the solid closed contour σ surrounding the branch cut. The BIC corresponds to the pole $s_p = -i\omega_a$ of $\hat{c}_a(s)$ on the imaginary axis internal to the branch cut, which is surrounded by two semi-circles whose radius R tends to zero.

bound states outside the continuum do not exist. For a given value of n_0, equation (22) is satisfied for a sufficiently small value of the coupling κ_a/κ_0 and provided that the frequency ω_a is not too close to the band edges. In the following, it will be assumed that no bound states exist outside the continuum.

(ii) *Bound states inside the continuum.* According to the results of Section 2.2, one BIC for the model expressed by equations (16) and (17) exists at $\Omega = \omega_a$ for any $n_0 \geq 2$, provided that the frequency ω_a of the discrete level assumes one of the following $(n_0 - 1)$ allowed values:

$$\omega_a = -2\kappa_0 \cos(m\pi/n_0) \quad (m = 1, 2, ..., n_0 - 1). \tag{23}$$

It should be pointed out that, from a physical viewpoint, the existence of BIC is not much related to the non-smoothness of the continuum in which the discrete state is embedded, as one might argue at first sight. The Van-Hove singularities in the density of states of the continuum at the band edges are related to the one dimensionality of the tight-binding model, however they are not relevant for the existence of BIC. In fact, for the same kind of continuum a constant coupling function $v(\omega) = \text{const.}$ would not lead to BIC, rather to two bound states outside the continuum, as shown in the Mahan's book [25]. So, what is the difference here? It is just the circumstance that in our model we have engineered the discrete-continuum spectral coupling (not the continuum itself) in such a way to suppress the interaction between the discrete level and one (or more) frequencies of the continuum. Of course, an open and basic question is how to implement such a colored interaction in a physical system, an issue which will be addressed in the next section.

(iii) *Decay dynamics and fractional decay due to BIC.* Suppose that H admits of one BIC but no bound states outside the continuum, i.e. that equations (22) and (23) are simultaneously satisfied. The decay law for $c_a(t)$ is given by the inverse Laplace transform equation (12). The Bromwich integration path in equation (12) can be deformed into the contour σ shown in Figure 2, where $\hat{c}_a(s)$ is always calculated on the first Riemannian sheet. The integral then comprises the pole contribution at $s_p = -i\omega_a$, which arises from the semi-circles surrounding the pole, and the principal-value integral of $[1/(2\pi i)]\hat{c}_a(s = -i\omega \pm 0^+)\exp(-i\omega t)$ along the interval $-2\kappa_0 < \omega < 2\kappa_0$ of the imaginary axis from the two sides $\mathrm{Re}(s) = \pm 0^+$ of the branch cut, i.e.

$$c_a(t) = c_{pole}(t) + c_d(t). \tag{24}$$

Using equations (12, 14, 19) and (21), after some straightforward calculations one then obtains:

$$c_{pole}(t) = \frac{\exp(-i\omega_a t)}{1 + \frac{n_0}{2}\left(\frac{\kappa_a}{\kappa_0}\right)^2 \left[1 - \left(\frac{\omega_a}{2\kappa_0}\right)^2\right]^{-1/2}} \tag{25}$$

for the pole contribution (non-decaying term), and

$$\begin{aligned} c_d(t) = {} & \frac{1}{2\pi}\left(\frac{\kappa_a}{\kappa_0}\right)^2 \mathcal{P}\int_0^\pi dk\ \sin^2(n_0 k)\exp(2i\kappa_0 t\cos k) \\ & \times\ \{[\omega_a/(2\kappa_0) + \cos k - (\kappa_a/2\kappa_0)^2 \sin(2n_0 k)/\sin k]^2 \\ & + [\kappa_a^2/(2\kappa_0^2)]^2 \sin^4(n_0 k)/\sin^2 k\}^{-1} \end{aligned} \tag{26}$$

for the decay term. The existence of a BIC is thus responsible for a fractional decay of the amplitude $c_a(t)$. Such a fractional decay is different from the most common one encountered in other single-level Fano-Anderson models (see, e.g., [39]) since in our model the fractional decay is due to the existence of a BIC. Note also that, if no bound states exist, i.e. if equation (22) is satisfied but one of the resonance conditions (23) is not satisfied, the amplitude $c_a(t)$ fully decays toward zero. In this case, one simply has $c_a(t) = c_d(t)$, where $c_d(t)$ is given again by equation (26) in which the principal value of the integral may be omitted. It is worth considering the limit $n_0 \to \infty$ for the model expressed by equations (16) and (17). In this case, in the integral on the second right term of equation (20) the rapidly-oscillating function $\sin^2(n_0 k)$ can be replaced by its cycle-averaged value $1/2$, i.e. one can approximately write

$$\Sigma(s) \simeq \frac{\kappa_a^2}{\pi}\int_0^\pi dk \frac{1}{is + 2\kappa_0 \cos k} = \frac{-i\kappa_a^2}{\sqrt{s^2 + 4\kappa_0^2}} \tag{27}$$

which from equation (14) yields

$$G(\omega) \simeq \frac{\kappa_a^2}{\pi\sqrt{4\kappa_0^2-\omega^2}} \tag{28}$$

and

$$\Delta(\omega) = \begin{cases} -\frac{\kappa_a^2}{\sqrt{\omega^2-4\kappa_0^2}} & \omega < -2\kappa_0 \\ 0 & -2\kappa_0 < \omega < 2\kappa_0 \\ \frac{\kappa_a^2}{\sqrt{\omega^2-4\kappa_0^2}} & \omega > 2\kappa_0. \end{cases} \tag{29}$$

Note that the above expressions for $G(\omega)$ and $\Delta(\omega)$ correspond to the limit of a tight-binding Fano-Anderson model with an uncolored interaction (see, e.g., Ref. [25], pp. 283–285), i.e. to a flat interaction function $v(k) \simeq \kappa_a/\sqrt{\pi}$. In this case it is known [25] that BIC do not exist at any value of ω_a, whereas two bound states outside the continuum are always found. The physical explanation of the disappearance of BIC in the $n_0 \rightarrow \infty$ limit will be discussed in the next section.

4 A tight-binding lattice realization of the colored Fano-Anderson model

In this section we propose a simple and noteworthy physical realization of the Fano-Anderson model with colored interaction discussed in the previous section, which may describe either electron or photon transport phenomena in condensed-matter or photonic tight-binding lattices. Since k varies in the range $0<k<\pi$, we can expand $c(k,t)$ as a Fourier series of sine terms solely according to

$$c(k,t) = -\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty} c_n(t)\sin(nk) \tag{30}$$

where the time-dependent coefficients c_n are given by

$$c_n(t) = -\sqrt{\frac{2}{\pi}}\int_0^{\pi} dk\, c(k,t)\sin(nk). \tag{31}$$

Taking into account that

$$\int_0^{\pi} dk\ \sin(nk)\sin(mk) = \frac{\pi}{2}\delta_{n,m} \tag{32}$$

($n,m \geq 1$), from equations (4, 16, 17, 30) and (31) the equations of motion for the coefficients c_n can be readily derived and read

$$i\dot{c}_n = -\kappa_0(c_{n+1}+c_{n-1}) - \kappa_a c_a \delta_{n,n_0} \quad (n \geq 2) \tag{33}$$

$$i\dot{c}_1 = -\kappa_0 c_2 - \kappa_a c_a \delta_{n_0,1}. \tag{34}$$

The equation for c_a (Eq. (3)) then reads

$$i\dot{c}_a = \omega_a c_a - \kappa_a c_{n_0}. \tag{35}$$

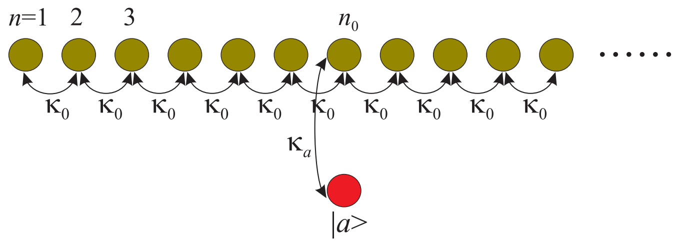

Fig. 3. Schematic of a localized state $|a\rangle$ coupled to a tight-binding semi-infinite lattice which realizes the Hamiltonian model (36). κ_0 is the hopping amplitude between adjacent sites in the lattice, whereas κ_a is the hopping amplitude between the localized state $|a\rangle$ and the site $|n_0\rangle$ of the lattice.

In the present form, equations (33–35) can be derived from the tight-binding Hamiltonian in the Wannier representation

$$H_{TB} = -\hbar\sum_{n=1}^{\infty}\kappa_0(|n\rangle\langle n+1| + |n+1\rangle\langle n|) + \hbar\omega_a|a\rangle\langle a| - \hbar\kappa_a(|a\rangle\langle n_0| + |n_0\rangle\langle a|) \tag{36}$$

which describes the interaction of the localized state $|a\rangle$ with the n_0-th site of a semi-infinite one-dimensional tight-binding lattice in the nearest-neighbor approximation (see Fig. 3). The tight-binding model expressed by equation (36) has been used to study transport phenomena in different physical systems, including photon tunneling dynamics in coupled optical waveguides [48] or in coupled photonic cavities [43,44,46,47], charge transfer of adatoms to a one-dimensional lattice of quantum dots [42], or decay of the polarization in spin chains [49]. For instance, equation (36) may describe charge transfer between an adatom localized state and a one-dimensional miniband associated with a quantum dot array, the adatom being attached to the semiconductor quantum-dot array surface [42]. It should be noted that these previous models considered either a semi-infinite tight-binding chain with a boundary defect [48–50] corresponding to the special case $n_0 = 1$, where however no BIC exist, or to an infinite array [25,42], i.e. to $n_0 \rightarrow \infty$ corresponding to a non-colored interaction, where again BIC do not exist. However, as shown in the previous section, for a finite value of n_0 larger than one, i.e. by considering a semi-infinite lattice in which a defect state interacts with a lattice site $|n_0\rangle$ near (but not at) the boundary, BIC at certain frequencies ω_a do exist according to equation (23). In the tight-binding representation (36) the existence of BIC has a simple physical explanation which is related to a *boundary effect* of the semi-infinite lattice: BIC correspond to localized states in the lattice with $c_n = 0$ for $n \geq n_0$. In fact, let us look for a solution to equations (33–35) of the form $c_n = \bar{c}_n\exp(-i\Omega t)$, $c_a = \bar{c}_a\exp(-i\Omega t)$, with $\bar{c}_n = 0$ for $n \geq n_0$. From equations (35) and (33) with $n = n_0$ it then follows that $\Omega = \omega_a$ and $\bar{c}_a = -(\kappa_0/\kappa_a)\bar{c}_{n_0-1}$, whereas after setting $\mathbf{c} \equiv (\bar{c}_1, \bar{c}_2, ..., \bar{c}_{n_0-1})^T$ from equations (34) and (33) with $n \leq n_0-1$ one obtains that $\mathbf{c}$ and Ω are the eigenvectors and corresponding eigenvalues of the $(n_0-1)\times(n_0-1)$ matrix

$$\mathcal{M}=\begin{pmatrix} 0 & -\kappa_0 & 0 & 0 & ... & 0 & 0 & 0 \\ -\kappa_0 & 0 & -\kappa_0 & 0 & ... & 0 & 0 & 0 \\ 0 & -\kappa_0 & 0 & -\kappa_0 & ... & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & ... & -\kappa_0 & 0 & -\kappa_0 \\ 0 & 0 & 0 & 0 & ... & 0 & -\kappa_0 & 0 \end{pmatrix}, \quad (37)$$

i.e. $\mathcal{M}\mathbf{c}=\omega_a\mathbf{c}$. Diagonalization of the matrix $\mathcal{M}$ yields for the eigenvalues the following expression

$$\Omega_m=-2\kappa_0\cos(m\pi/n_0) \ \ (m=1,2,...,n_0-1) \quad (38)$$

with corresponding eigenvectors $\bar{c}_n^{(m)}=\sin(m\pi n/n_0)$. Note that the values of Ω_m given by equation (38) are precisely the resonance frequencies for the existence of BIC found in the previous section (see Eq. (23)). Therefore in the tight-binding realization of the colored Fano-Anderson model BIC arise due to a trapping effect which localizes the excitation between the boundary of the semi-infinite chain and the $|n_0-1\rangle$th site of the chain. The coupling of the state $|a\rangle$ with the lattice site $|n_0\rangle$ allows for the vanishing of the amplitudes c_n at lattice sites $n\geq n_0$ via quantum destructive interference. It should be noted that similar trapping mechanisms supporting BIC in tight-binding models have been recently found in triple or quadruple dot molecules connected to two leads [20,22], which are modeled as two semi-infinite tight-binding lattices. In these models, BIC correspond to vanishing of the wave function at the sites in the molecule in contact with the leads, i.e. BIC are fully localized in the molecule sites but not in the leads. Conversely, the present tight-binding model (Eq. (36)) involves solely one localized state side-coupled to a semi-infinite lattice, and therefore BIC can not simply correspond to a decoupling of the localized state with the lattice. This is clearly demonstrated by the fact that the wave function of a BIC for the model expressed by equation (36) is non-vanishing *even* in a portion of the lattice (from the boundary site $|1\rangle$ to the site $|n_0-1\rangle$). Additionally, BIC cease to exist as $n_0\to\infty$, i.e. lattice truncation is essential to sustain BIC.

We checked the existence of BIC induced by this trapping mechanism by a direct numerical analysis of the coupled mode equations (33–35) using a fourth-order variable-step Runge-Kutta method with smoothly absorbing boundary conditions at the right boundary of the lattice to avoid spurious reflections due to truncation of equation (33). As an example, Figues 4 and 5 show the decay dynamics of $|c_a(t)|$ for parameter values corresponding to a complete decay (Fig. 4), i.e. to the absence of a BIC, and to a fractional decay (Fig. 5) related to the existence of a BIC. In the figures, the dynamical evolution of the lattice site amplitudes $|c_n(t)|$ is also depicted on a gray-scale plot, showing either a diffusion (Fig. 4b) or a localization (Fig. 5b) of the excitation transferred from the localized state $|a\rangle$ to the lattice site $|n_0\rangle$. We checked that the numerically-computed decay law for $c_a(t)$ exactly reproduces the curve predicted by the analytical decay law equations (24–26). Note that, as the resonance condition (23) for the existence of BIC is satisfied, fractional decay of c_a is attained (Fig. 5a), which clearly corresponds to trapping of the excitation at the lattice sites $|1\rangle,|2\rangle,...,|n_0-1\rangle$ with a destructive interference of site excitation for $n\geq n_0$ (Fig. 5b). Conversely, for a frequency ω_a which does not satisfy the resonance condition (23) for some integer m, the amplitude $c_a(t)$ fully decays toward zero (Fig. 4a) and the excitation transferred to the lattice diffuses along the lattice without being trapped (Fig. 4b).

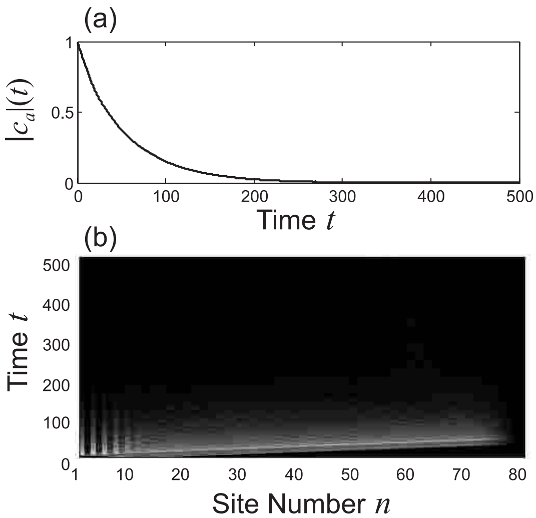

Fig. 4. (a) Decay dynamics of the amplitude $|c_a(t)|$ as obtained by numerical analysis of equations (33–35) for parameter values $\kappa_0=1$, $\kappa_a=0.2$, $n_0=12$, and for $\omega_a=0.15$, corresponding to the absence of BIC. In (b) the temporal evolution of the amplitudes $|c_n(t)|$ at the lattice sites is also shown on a grey-scale plot.

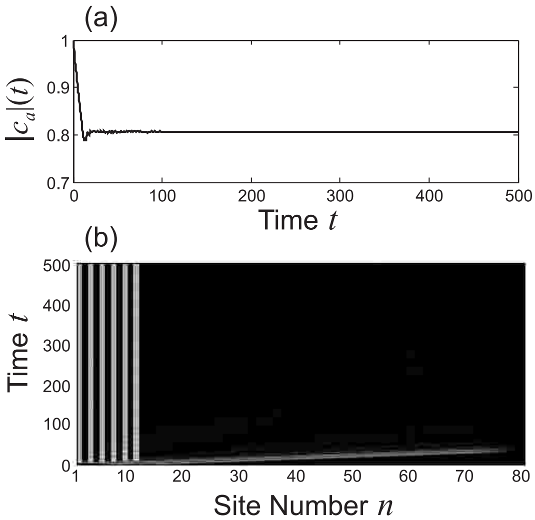

Fig. 5. Same as Figure 4, but for $\omega_a=0$, corresponding to the existence of one BIC.

5 Conclusions

In this work an exactly-solvable single-level Fano-Anderson model which admits of bound states inside a structured continuum has been proposed, and its relevance to tight-binding lattice models generally adopted to study electron or photon transport phenomena in condensed-matter or photonic systems has been discussed. As previously proposed models supporting BIC in single-level Fano-Anderson models require point-like gaps in the density of states [31,40,41] — a condition which does not seem to have simple physical realizations [31] — in the present work it has been shown that BIC can exist in a tight-binding structured continuum without point-like gaps provided that the interaction of the localized state with the continuum is engineered in such a way to suppress discrete-continuum coupling at some frequencies inside the band. A lattice realization for such a colored Fano-Anderson model, which may be of relevance to model photon or electron transport in certain photonic or condensed-matter systems [42–50], has been proposed, and a simple physical explanation of the existence of BIC as a trapping effect sustained by lattice truncation has been highlighted.

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